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A q-analog of Freudenthal's weight multiplicity formula

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1. INTRODUCTION

In this paper, we give a formula which calculates certain of Lusztig's q-analogs of weight multiplicity [7] and hence certain Kazhdan-Lusztig polynomials for the affine Weyl group of a semisimple, simply connected algebraic group G over $\mathbb{C}$. It does so recursively with respect to the standard partial ordering on the character module of a maximal torus of G.

Fix a maximal torus T of G, and denote by $X^* = X^*(T)$ and $X_* = X_*(T)$ the character and co-character modules of T, respectively. Furthermore, let $\langle\ ,\ \rangle : X^* \times X_* \longrightarrow \mathbb{Z}$ denote the standard bilinear pairing between X^* and X_*. Denote by Φ the set of roots of T and by W the corresponding Weyl group. Choose a subset $\Phi^+ \subset \Phi$ of positive roots. Corresponding to Φ^+, we have a set $X^+ \subset X^*$ of dominant characters and a partial ordering $\leq$ on X^*. We denote the element $\frac{1}{2}\sum_{\alpha\in\Phi^+} \alpha$ of X^* by ρ. Similarly, we define $\rho^\vee$ in $X_*(T)$ by $\frac{1}{2}\sum_{\alpha\in\Phi^+} \alpha^\vee$, where $\alpha^\vee$ is the coroot corresponding to the root α. Let Λ be the submodule of X^* generated by Φ. Denote by W_{af} the affine Weyl group $\Lambda \rtimes W$ and by $\tilde{W}_{\mathrm{af}}$ the extended affine Weyl group $X^* \rtimes W$.

Let $\mathbb{Z}[X^*][[q]]$ be the ring of power series in q over the group ring $\mathbb{Z}[X^*]$. Define the polynomial $\mathcal{P}(\lambda)$ in $\mathbb{Z}[q]$ by the equation

$$\sum_{\lambda\in X^*} \mathcal{P}(\lambda)e(\lambda) = \prod_{\alpha\in\Phi^+} (1 - e(\alpha)q)^{-1}$$

in $\mathbb{Z}[X^*][[q]]$. Then for μ in X^*, $\lambda \in X^+$, Lusztig defined a q-analog of weight multiplicity $\mathcal{M}_\lambda^\mu$ by

$$\mathcal{M}_\lambda^\mu(q) = \sum_{\sigma \in W} \varepsilon(\sigma)\mathcal{P}(\sigma(\lambda + \rho) - (\mu + \rho)),$$

where $\varepsilon(\sigma)$ is the determinant of σ acting on X^*. We observe that $\mathcal{M}_\lambda^\mu(1) = m_{\mu,\lambda}$ gives Kostant's expression for the dimension of the μ-weight space of the irreducible representation V_λ of G of highest weight λ. We also note that $\mathcal{M}_\lambda^\mu(q) \neq 0$ if and only if $\lambda > \mu$. For $\lambda, \mu \in X^+$, $\lambda > \mu$, define $P_{\mu,\lambda}(q) = q^{\langle \lambda - \mu, \rho^\vee \rangle}\mathcal{M}(q^{-1})$. Then $P_{\mu,\lambda}(q)$ is in $\mathbb{Z}[q]$ since the character $\lambda - \mu$ is a sum of roots and $\sigma(\lambda + \rho) - (\mu + \rho) \leq \lambda - \mu$. Also $P_{\mu,\lambda}(1) = m_{\mu,\lambda}$.

For $\lambda \in X^+$ let w_λ be the element of $\tilde{W}_{\text{af}}$ of maximal length in the double coset $W\lambda W$. If $\mu \leq \lambda$ in X^+ then $w_\mu \leq w_\lambda$ under the Bruhat ordering on $\tilde{W}$ corresponding to Φ^+. Thus, we may consider the Kazhdan-Lusztig polynomial $P_{w_\mu, w_\lambda}(q)$ (see [5], [6]) for W_{af} associated to w_μ and w_λ. Lusztig ([7]) proved that such Kazhdan-Lusztig polynomials arise as matrix coefficients of the inverse of the Satake transform. Lusztig then conjectured and Kato ([4, Theorem 1.8]) later proved in general that $P_{\mu,\lambda} = P_{w_\mu, w_\lambda}$. The polynomials $P_{\mu,\lambda}$ can be shown to have non-negative coefficients and to be of degree strictly less than $\langle \lambda - \mu, \rho^\vee \rangle$ by the general theory of Kazhdan-Lusztig polynomials ([7]).

We will derive a formula which computes certain $\mathcal{M}_\lambda^\mu$ (and hence certain $P_{\mu,\lambda}$) recursively, avoiding the summation over W which appears in the q-analog of Kostant's formula since the size of W grows rather quickly as the rank of G increases. Let us first consider the case $q = 1$.

For λ in X^+, $\mu \leq \lambda$, there is a well-known recursive formula for the multiplicity $m_{\mu,\lambda} = \mathcal{M}_\lambda^\mu(1)$ of the weight μ in the representation V_λ due to Freudenthal. Let $(\ ,\ )$ be an inner product on X^* which is W-invariant and let $\|\mu\| = (\mu, \mu)^{1/2}$ for μ in X^*. Freudenthal's multiplicity formula states that for any μ, λ in X^*

$$(1) \qquad (\|\lambda + \rho\|^2 - \|\mu + \rho\|^2)m_{\mu,\lambda} = 2 \sum_{\alpha \in \Phi^+} \sum_{i \geq 1} (\alpha, \mu + i\alpha)m_{\mu + i\alpha, \lambda}.$$

Note that the formula yields $m_{\mu,\lambda}$ as a linear combination of the (finitely many non-zero) $m_{\nu,\lambda}$ with $\nu > \mu$. The q-analog of this formula is the following.

Proposition 1.1. *Suppose $\mu \in X^*, \lambda \in X^+$. Then,*

$$\begin{aligned}(\|\lambda + \rho\|^2 - \|\mu + \rho\|^2)&\mathcal{M}_\lambda^\mu(q) \\ &= 2 \sum_{\alpha \in \Phi^+} \sum_{i \geq 1} (\alpha, \mu + i\alpha)q^i \mathcal{M}_\lambda^{\mu + i\alpha}(q) \\ &\quad + \sum_{\substack{\alpha, \beta \in \Phi^+ \\ \alpha \neq \beta}} \sum_{i \geq 1} \sum_{j \geq 0} (\alpha, \beta) q^{i+j} \mathcal{M}_\lambda^{\mu + i\alpha + j\beta}(q).\end{aligned}$$

In case $\mu, \lambda \in X^+$, if one substitutes q^{-1} for q and then $q^{\langle \mu - \lambda, \rho^\vee \rangle} P_{\mu,\lambda}(q)$ for $\mathcal{M}_\lambda^\mu(q^{-1})$ one obtains the analogous result for Kazhdan-Lusztig polynomials.

Corollary 1.2. *Suppose $\mu, \lambda \in X^+$. Then,*

$$
\begin{aligned}
&\left(\|\lambda+\rho\|^2-\|\mu+\rho\|^2\right)P_{\mu,\lambda}(q)\\
&\quad=2\sum_{\alpha\in\Phi^+}\sum_{i\geq 1}(\alpha,\mu+i\alpha)q^{i(\langle\alpha,\rho^\vee\rangle-1)}P_{\mu+i\alpha,\lambda}(q)\\
&\quad\quad+\sum_{\alpha\neq\beta}\sum_{i\geq 1}\sum_{j\geq 0}(\alpha,\beta)q^{i(\langle\alpha,\rho^\vee\rangle-1)+j(\langle\beta,\rho^\vee\rangle-1)}P_{\mu+i\alpha+j\beta,\lambda}(q).
\end{aligned}
$$

Note that if one sets $q=1$ in either formula, then the first term on the right becomes the right side of (1), while the second term on the right becomes a sum over pairs of distinct positive roots which, in agreement with Freudenthal's formula, can be shown to vanish by simple computations involving the various root systems of rank 2.

Remark. The formula in Proposition 1.1 computes the polynomials $\mathcal{M}_\lambda^\mu(q)$ in essentially the same fashion that Freudenthal's formula computes the weight multiplicities $m_{\mu,\lambda}=\mathcal{M}_\lambda^\mu(1)$. However the q-analog is applicable in fewer cases due to the following. When λ and μ satisfy $\|\lambda+\rho\|=\|\mu+\rho\|$, both formulae yield no information. This is not a problem when one wishes merely to compute weight multiplicities since this condition on λ and μ implies that either $\lambda=\mu$ or $\|\lambda\|<\|\mu\|$. In the former case, one knows that $m_{\mu,\lambda}=1$, and in the latter, $m_{\mu,\lambda}=0$, by the representation theory of algebraic groups.

Now $\mathcal{M}_\lambda^\lambda(q)$ is also always equal to 1. However, for $\mu<\lambda$, $\mathcal{M}_\lambda^\mu(q)$ is not necessarily equal to 0 when $\|\lambda\|<\|\mu\|$. Of course, if μ and λ are dominant, then $\|\lambda\|<\|\mu\|$ implies that $\mu\not\leq\lambda$, so that in this instance, we can deduce that $\mathcal{M}_\lambda^\mu(q)=0$. However, even when one restricts one's attention to the $\mathcal{M}_\lambda^\mu(q)$ for μ,λ in X^+ (the $\mathcal{M}_\lambda^\mu(q)$ relevant to the Satake isomorphism are of this type), it is likely that one would still have to compute certain $\mathcal{M}_\lambda^\nu(q)$ for ν not in X^+ because of the nature of the recursion. It is therefore possible that in order to compute $\mathcal{M}_\lambda^\mu(q)$ with the formula one may need to determine a polynomial $\mathcal{M}_\lambda^\nu(q)$ about which the q-analog yields no information. For example, $\mathcal{M}_\lambda^0$ is almost never computable using this q-analog alone. But as long as there are no characters ν (outside of X^+) such that $\mu<\nu<\lambda$ and $\|\lambda+\rho\|=\|\mu+\rho\|$ (as is the case, for example, when $\lambda,\mu\in X^+$ and λ is sufficiently close to μ in relation to its distance from the boundary of the Weyl chamber corresponding to X^+), recursive applications of the formula will explicitly yield $\mathcal{M}_\lambda^\mu(q)$.

For μ and λ as in the above paragraph our q-analog of Freudenthal's formula is in general more practical than the q-analog of Kostant's formula. Also, for computing the Kazhdan-Lusztig polynomials corresponding to such μ and λ our formula is more efficient than the recursive formula with respect to the Bruhat ordering on W_{af}, which typically involves summation over a large number of elements of W_{af}. It should nevertheless be noted that Broer's iterative method of calculating Lusztig's q-analogs in [1, §4] involves fewer computations than this formula and is applicable in all cases. Broer's method was implemented in a computer program by Stembridge using his *Maple* packages *coxeter* and *weyl* [8].

The proof of Proposition 1.1 will follow along the lines of the standard derivation of Freudenthal's formula from Kostant's (see [2, §25.2]), employing formal power series in $\mathbb{Z}[X^*][[q]]$. We will denote the standard basis elements of $\mathbb{Z}[X^*]$ over $\mathbb{Z}$ by $e(\mu)$ (μ in X^*).

Fix the character λ in X^+ and form the sum

$$C(q) = \sum_{\mu \in X^*} \mathcal{M}_\lambda^\mu(q) e(\mu).$$

Setting $\nu = \sigma(\lambda+\rho) - (\mu+\rho)$, we see that $C(q)$ factors as

$$\begin{aligned}
\sum_{\mu \in X^*} \mathcal{M}_\lambda^\mu(q) e(\mu) &= \sum_{\mu \in X^*} \sum_{\sigma \in W} \varepsilon(\sigma) \mathcal{P}(\sigma(\lambda+\rho) - (\mu+\rho)) e(\mu) \\
&= \sum_{\nu \in X^*} \sum_{\sigma \in W} \varepsilon(\sigma) \mathcal{P}(\nu) e(\sigma(\lambda+\rho) - (\nu+\rho)) \\
&= \left(\sum_{\sigma \in W} \varepsilon(\sigma) e(\sigma(\lambda+\rho)) \right) \cdot \left(e(-\rho) \sum_{\nu \in X^*} \mathcal{P}(\nu) e(-\nu) \right) \\
&= \left(\sum_{\sigma \in W} \varepsilon(\sigma) e(\sigma(\lambda+\rho)) \right) \cdot \frac{e(-\rho)}{\prod_{\alpha \in \Phi^+} (1 - q e(-\alpha))}
\end{aligned} \tag{2}$$

where the last equality follows from the definition of $\mathcal{P}$. Let $M(q)$ be the first factor in (2) and $N(q) = e(\rho) \prod_{\alpha \in \Phi^+} (1 - q e(-\alpha))$ the inverse of the second factor so that $M(q) = N(q) C(q)$.

Let

$$\triangle : \mathbb{Z}[X^*][[q]] \longrightarrow \mathbb{Z}[X^*][[q]]$$

be the map defined by

$$\triangle(e(\mu) q^i) = (\mu, \mu) e(\mu) q^i$$

for μ in X^*. As in [2, §25.2], we will apply this operator to $M(q)$ and $N(q)C(q)$, and equating the resulting elements of $\mathbb{Z}[X^*][[q]]$ will yield the desired q-analog of Freudenthal's formula.

It is clear that

$$\triangle (M(q)) = \|\lambda+\rho\|^2 M(q) = \|\lambda+\rho\|^2 N(q) C(q) \tag{3}$$

as $(\ ,\ )$ is Weyl-invariant. We now turn our attention to calculating the term $\triangle(N(q)C(q))$. Our first step will be to give a formula for the effect of $\triangle$ on a product of elements of $\mathbb{Z}[q^{1/2}, q^{-1/2}]((X^*))$.

We define

$$\nabla : \mathbb{Z}[X^*][[q]] \longrightarrow X^* \otimes \mathbb{Z}[X^*][[q]]$$

by the formula

$$\nabla(e(\mu)) = \mu \otimes e(\mu).$$

Also we define a bilinear pairing on $X^* \otimes \mathbb{Z}[X^*][[q]]$ taking values in $\mathbb{Z}[X^*][[q]]$ (also denoted by (,)) by

$$(\lambda_1 \otimes e(\mu_1)q^{i_1}, \lambda_2 \otimes e(\mu_2)q^{i_2}) = (\lambda_1, \lambda_2)e(\mu_1 + \mu_2)q^{i_1+i_2}.$$

Then if $f, g \in \mathbb{Z}[X^*][[q]]$ we have (see [2, 25.18])

$$\triangle(fg) = \triangle(f)g + 2(\nabla(f), \nabla(g)) + f \triangle(g). \tag{4}$$

Furthermore, ∇ satisfies Leibnitz's rule, namely

$$\nabla(fg) = \nabla(f)g + f\nabla(g).$$

We now apply (4) to $N(q)C(q)$. We find

$$\triangle(N(q)C(q)) = \triangle(N(q))C(q) + 2(\nabla(N(q)), \nabla(C(q))) + N(q)\triangle(C(q)).$$

This, together with (3) implies that

$$\begin{aligned} &\|\lambda+\rho\|^2 C(q) - \triangle(C(q)) \\ &\quad = 2N(q)^{-1}(\nabla(N(q)), \nabla(C(q))) + N(q)^{-1}\triangle(N(q))C(q). \end{aligned} \tag{5}$$

Clearly, $\triangle(C(q)) = \sum_{\mu \in X^*} \|\mu\|^2 \mathcal{M}_\lambda^\mu(q)e(\mu)$. We must now compute the two terms on the right of (5).

First let us define for $S \subset \Phi^+$

$$Q_S = \prod_{\gamma \in \Phi^+ \setminus S} (1 - qe(-\gamma)).$$

Note that $N(q) = e(\rho)Q_\emptyset$ and $Q_\emptyset^{-1}Q_S = \prod_{\gamma \in S}(1 - qe(-\gamma))^{-1}$. Then using the generalization to multiple factors of the Leibnitz rule for ∇, we compute the first of the two terms on the right of (5)

$$\begin{aligned} &2N(q)^{-1}(\nabla(N(q)), \nabla(C(q))) \\ &\quad = 2e(-\rho)Q_\emptyset^{-1} \cdot \\ &\quad \left(\rho \otimes e(\rho)Q_\emptyset + \sum_{\alpha \in \Phi^+} \alpha \otimes qe(\rho)e(-\alpha)Q_{\{\alpha\}}, \sum_{\mu \in X^*} \mu \otimes \mathcal{M}_\lambda^\mu(q)e(\mu)\right) \\ &\quad = 2\sum_{\mu \in X^*} (\rho, \mu)\mathcal{M}_\lambda^\mu(q)e(\mu) \\ &\qquad + 2\sum_{\alpha \in \Phi^+}\sum_{\mu \in X^*} (\alpha, \mu)q\mathcal{M}_\lambda^\mu(q)e(\mu - \alpha)(1 - qe(-\alpha))^{-1} \\ &\quad = 2\sum_{\mu \in X^*} (\rho, \mu)\mathcal{M}_\lambda^\mu(q)e(\mu) \\ &\qquad + 2\sum_{\alpha \in \Phi^+}\sum_{\mu \in X^*}\sum_{i \geq 0} (\alpha, \mu)q^{i+1}\mathcal{M}_\lambda^\mu(q)e(\mu - (i+1)\alpha) \\ &\quad = \sum_{\mu \in X^*} (2\rho, \mu)\mathcal{M}_\lambda^\mu(q)e(\mu) \\ &\qquad + 2\sum_{\mu \in X^*}\sum_{\alpha \in \Phi^+}\sum_{i \geq 1} (\alpha, \mu + i\alpha)q^i\mathcal{M}_\lambda^{\mu+i\alpha}(q)e(\mu). \end{aligned} \tag{6}$$

For the second term on the right of (5), we require the generalization of (4) to n factors

$$\triangle(f_1 \cdots f_n) = \sum_{k=1}^{n} \triangle(f_k) \prod_{i \neq k} f_i + \sum_{j \neq k} (\nabla f_j, \nabla f_k) \prod_{i \neq j,k} f_i.$$

Applying this formula to $N(q)^{-1} \triangle (N(q))C(q)$ yields

$$\begin{aligned} e(-\rho)Q_\emptyset^{-1}C(q)\cdot\Bigg[\|\rho\|^2 e(\rho)Q_\emptyset &+ \sum_{\alpha\in\Phi^+} e(\rho)\left(-\|\alpha\|^2 qe(-\alpha)\right)Q_{\{\alpha\}} \\ &+ 2\sum_{\alpha\in\Phi^+} (\nabla(e(\rho)), \nabla(1-qe(-\alpha)))Q_{\{\alpha\}} \\ &+ \sum_{\alpha\neq\beta} e(\rho)\,(\nabla(1-qe(-\alpha)), \nabla(1-qe(-\beta)))\,Q_{\{\alpha,\beta\}}\Bigg]. \end{aligned}$$

Computing the action of ∇, we obtain

$$\begin{aligned} &\|\rho\|^2 C(q) - C(q)\sum_{\alpha\in\Phi^+} \|\alpha\|^2 qe(-\alpha)\,(1-qe(-\alpha))^{-1} \\ &+ 2e(-\rho)C(q)\sum_{\alpha\in\Phi^+} (\rho\otimes e(\rho), \alpha\otimes qe(-\alpha))\,(1-qe(-\alpha))^{-1} \\ &+ C(q)\sum_{\alpha\neq\beta} (\alpha\otimes qe(-\alpha), \beta\otimes qe(-\beta))(1-qe(-\alpha))^{-1}(1-qe(-\beta))^{-1} \\ &\quad = \|\rho\|^2 C(q) - C(q)\sum_{\alpha\in\Phi^+}\sum_{i\geq 0} \|\alpha\|^2 q^{(i+1)} e(-(i+1)\alpha) \\ &\qquad + 2C(q)\sum_{\alpha\in\Phi^+}\sum_{i\geq 0} (\rho,\alpha) q^{(i+1)} e(-(i+1)\alpha) \\ &\qquad + C(q)\sum_{\alpha\neq\beta}\sum_{i,j\geq 0} (\alpha,\beta) q^{(i+1)+(j+1)} e(-(i+1)\alpha-(j+1)\beta). \end{aligned}$$

Substituting the definition of $C(q)$ into this last expression and shifting the summations over i, j, and μ appropriately, we obtain that $N(q)^{-1} \triangle (N(q))C(q)$ equals

$$\begin{aligned} &\|\rho\|^2 \sum_{\mu\in X^*} \mathcal{M}_\lambda^\mu(q)e(\mu) - \sum_{\mu\in X^*}\sum_{\alpha\in\Phi^+}\sum_{i\geq 1} (\alpha,\alpha)q^i \mathcal{M}_\lambda^{\mu+i\alpha}(q)e(\mu) \\ (7)\qquad &+ \sum_{\mu\in X^*}\sum_{\alpha\in\Phi^+}\sum_{i\geq 1} (2\rho,\alpha)q^i \mathcal{M}_\lambda^{\mu+i\alpha}(q)e(\mu) \\ &+ \sum_{\mu\in X^*}\sum_{\alpha\neq\beta}\sum_{i,j\geq 1} (\alpha,\beta)q^{i+j}\mathcal{M}_\lambda^{\mu+i\alpha+j\beta}(q)e(\mu). \end{aligned}$$

Now we insert (6), (7) and the expression for $\triangle(C(q))$ into (5), equate the coefficients of $e(\mu)$ for each μ in X^*, and group the terms containing $\mathcal{M}_\lambda^\mu$ on the left. We obtain that for every μ in X^*

$$
\begin{aligned}
&\big(\|\lambda+\rho\|^2-\|\mu\|^2-\|\rho\|^2-(2\rho,\mu)\big)\mathcal{M}_\lambda^\mu(q)\\
&\quad=-\sum_{\alpha\in\Phi^+}\sum_{i\geq 1}(\alpha,\alpha)q^i\mathcal{M}_\lambda^{\mu+i\alpha}(q)\\
&\qquad+\sum_{\alpha\in\Phi^+}\sum_{i\geq 1}(2\rho,\alpha)q^i\mathcal{M}_\lambda^{\mu+i\alpha}(q)\\
&\qquad+\sum_{\alpha\neq\beta}\sum_{i,j\geq 1}(\alpha,\beta)q^{i+j}\mathcal{M}_\lambda^{\mu+i\alpha+j\beta}(q)\\
&\qquad+2\sum_{\alpha\in\Phi^+}\sum_{i\geq 1}(\alpha,\mu+i\alpha)q^i\mathcal{M}_\lambda^{\mu+i\alpha}(q).
\end{aligned}\tag{8}
$$

Since $2\rho=\sum_{\beta\in\Phi^+}\beta$, the sum of the first two terms on the right of (8) equals

$$
\begin{aligned}
&\sum_{\alpha\in\Phi^+}\sum_{i\geq 1}(2\rho-\alpha,\alpha)q^i\mathcal{M}_\lambda^{\mu+i\alpha}(q)\\
&\quad=\sum_{\alpha\in\Phi^+}\sum_{i\geq 1}\left(\sum_{\beta\in\Phi^+}\beta-\alpha,\alpha\right)q^i\mathcal{M}_\lambda^{\mu+i\alpha}(q)\\
&\quad=\sum_{\alpha\neq\beta}\sum_{i\geq 1}(\alpha,\beta)q^i\mathcal{M}_\lambda^{\mu+i\alpha}(q).
\end{aligned}
$$

Also, we have

$$\|\mu+\rho\|^2=\|\mu\|^2+\|\rho\|^2+(2\rho,\mu)$$

so the coefficient in front of $\mathcal{M}_\lambda^\mu(q)$ is

$$\|\lambda+\rho\|^2-\|\mu+\rho\|^2.$$

Using these facts, (8) simplifies as

$$
\begin{aligned}
&\big(\|\lambda+\rho\|^2-\|\mu+\rho\|^2\big)\mathcal{M}_\lambda^\mu(q)\\
&\quad=\sum_{\alpha\neq\beta}\sum_{i\geq 1}(\alpha,\beta)q^i\mathcal{M}_\lambda^{\mu+i\alpha}(q)\\
&\qquad+\sum_{\alpha\neq\beta}\sum_{i,j\geq 1}(\alpha,\beta)q^{i+j}\mathcal{M}_\lambda^{\mu+i\alpha+j\beta}(q)\\
&\qquad+2\sum_{\alpha\in\Phi^+}\sum_{i\geq 1}(\alpha,\mu+i\alpha)q^i\mathcal{M}_\lambda^{\mu+i\alpha}(q)\\
&\quad=\sum_{\alpha\neq\beta}\sum_{i\geq 1}\sum_{j\geq 0}(\alpha,\beta)q^{i+j}\mathcal{M}_\lambda^{\mu+i\alpha+j\beta}(q)\\
&\qquad+2\sum_{\alpha\in\Phi^+}\sum_{i\geq 1}(\alpha,\mu+i\alpha)q^i\mathcal{M}_\lambda^{\mu+i\alpha}(q).
\end{aligned}
$$

This proves Proposition 1.1.

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